Flat Systems in Discrete Signal Processing

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ABSTRACT

The concept of flatness is here introduced for dynamic discrete-time systems analogously to the flatness of continuous-time systems. This concept gives a way for open-loop as well as closed-loop control design for dynamic systems when the goal is to drive the system from one steady-state to another. The successive derivatives of the so-called flat output and the control of a continuous-time system are substituted by their backward shifts in discrete approach. Some flatness based properties are preliminarily studied via a linear example. Relations to dead-beat control are also pointed out

1. INTRODUCTION

In many dynamical control and signal processing systems an intermediate goal is to drive the output of a system from one steady-state to another as quickly as possible. The recently coined and studied concept of flatness of nonlinear as well as linear differential equations and systems points out a way for straightforward open-loop control design. Differential flatness has been developed in the works of Michel Fliess and his co-workers in France, see, *e.g.*, [7], [8], [10], [12]. It has its origin in the beginning of 1900s in the studies of Elie Cartan on underdetemined differential systems, i.e. on the sets of differential equations (without control) having a lesser number of equations than variables. Flatness issues have also been studied from another viewpoint by utilizing differential forms and exterior algebras, *c.f.* Murray and co-workers in [20], and [21], again originating in the work of Cartan.

If the differential system is flat, its input and state can be expressed as functions of another variable, called a flat output, and of a finite number of its time derivatives, and *vice versa* this another variable can be respresented as a function of the state and control and of devivatives of the control. Then, for control design, starting from a desired flat output, one can construct the actual state (and output) and the open-loop control, which produces the output. This can be done easily just by differentiating sufficiently many times the flat output and by using the known, system dependent functions, see some design examples in [11].

Definition (Differential flatness) [10]. Consider a nonlinear ordinary differential system

$$\frac{dx}{dt} = f(x,u); \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \tag{1}$$

where x and u denote the state and the control of the system, respectively. If there exist algebraic functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and finite integers α, β, and γ such that for any (sufficiently differentiable) pair (x,u) satisfying the dynamics (1) there exists a vector-valued sufficiently differentiable function z ($z(t) \in \mathbb{R}^m$) satisfying

$$\begin{aligned} x &= \mathcal{A}(z, \dot{z}, \ldots, z^{(\alpha)}) \\ u &= \mathcal{B}(z, \dot{z}, \ldots, z^{(\beta)}) \\ z &= \mathcal{C}(x, u, \dot{u} \ldots, u^{(\gamma)}) \end{aligned} \tag{2}$$

then the system (1) is called *differentially flat* and the variable z is called a *flat, or linearizing output.*

Remark. It has to be noted that the flatness concept actually does not include the output, say y, at all. Then, in fact, the inclusion of the output equation for the considerations is, in principle, unnecessary. However, in any practical control problem there is an output to be controlled.

In this paper the discrete-time flatness is introduced according to Fliess & Marquez [13], which is based on the original study of Fliess [6]. Here it is demonstrated that minimal linear state-variable representations describe flat systems. A scalar example is given to illustrate open-loop control design based on flatness. Corresponding feedback control and nonlinear problems are discussed, too.

2. DIFFERENCE FLATNESS

There are two possibilities to extend the concept of flatness to discrete-time systems by using a definition analogous to that of differential flatness. The derivatives can be substituted by forward shifts or backward shifts, *i.e.* the term, say, $z(t+i) = q^i z(t)$ substitutes the derivative $\frac{d^i z(t)}{dt^i}$ or $z(t-i) = q^{-i}z(t)$ substitutes $\frac{d^i z(t)}{dt^i}$. Forward shifts were discussed, however, without further explicit constructions in Aranda-Bricaire *et al.* [1], p. 2016. Here we apply the following backward shift definition.

Definition (Difference flatness). Consider a nonlinear ordinary difference system

$$x(t+1) = f(x(t), u(t)); \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m \tag{3}$$

If there exist algebraic functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and finite integers α, β, and γ such that for any pair (x, u) satisfying the dynamics (3) there exists a vector-valued function z ($z(t) \in \mathbb{R}^m$) satisfying

$$\begin{aligned} x &= \mathcal{A}(z, q^{-1}z \ldots, q^{-\alpha}z) \\ u &= \mathcal{B}(z, q^{-1}z, \ldots, q^{-\beta}z) \\ z &= \mathcal{C}(x, u, q^{-1}u, \ldots, q^{-\gamma}u) \end{aligned} \tag{4}$$

then the system (3) is called *differencely flat* and the variable z is called a *flat, or linearizing output.*

2.1. Flatness in Linear SISO-Systems

Here we study a class of linear single-input - single-output (SISO) systems, which are controllable and observable, and which have the polynomial representation of the form ($a_n \neq 0$)

$$A(q^{-1})y(t) = B(q^{-1})u(t) \tag{5}$$

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + a_2 q^{-2} + \ldots + a_n q^{-n} \\ B(q^{-1}) &= b_1 q^{-1} + b_2 q^{-2} + \ldots + b_n q^{-n} \end{aligned}$$

Theorem 1. Linear difference systems of the form (5), where $A(q^{-1})$ and $B(q^{-1})$ are coprime, are differencely flat when represented in the controllable and observable form

$$x(t+1) = Fx(t) + Gu(t) \tag{6}$$

$$y(t) = Cx(t). \tag{7}$$

A flat output is defined, see [13], by

$$z(t) = S(q^{-1})y(t) + R(q^{-1})u(t). \tag{8}$$

where S and R satisfy, due to coprimeness of A and B, Bezout's equation

$$R(q^{-1})A(q^{-1}) + S(q^{-1})B(q^{-1}) = 1. \tag{9}$$

Furthermore, the flat output gives

$$u(t) = A(q^{-1})z(t) \tag{10}$$

$$y(t) = B(q^{-1})z(t) \tag{11}$$

The proof is omitted.

Remark. For a practical trajectory design the input-output description with its flatness defining equations (8)-(11) are a feasible way to proceed instead of using the state variable representation.

3. FLATNESS IN LINEAR MULTIVARIABLE SYSTEMS

Controllable and observable linear multivariable systems has two equivalent polynomial matrix fraction representations, see [5], p. 599. The left coprime fraction representation resembles the representation of a SISO system. The input-output system having the control $u(t) \in \mathbb{R}^m$, and the output $y(t) \in \mathbb{R}^k$ has a representation of the form

$$A(q^{-1})y(t) = B(q^{-1})u(t)$$

For the present author this representation did not open the way for flatness.

The *right coprime fraction representation* of the system transfer matrix is of the form ($D_n \neq 0$)

$$T(q^{-1}) = N(q^{-1})[D(q^{-1})]^{-1} \tag{12}$$

$$D(q^{-1}) = I + D_1 q^{-1} + \ldots + D_n q^{-n} \tag{13}$$

$$N(q^{-1}) = N_1 q^{-1} + N_2 q^{-2} + \ldots + N_n q^{-n} \tag{14}$$

where the matrix coefficients are $D_i \in \mathbb{R}^{m\times m}$ and $N_i \in \mathbb{R}^{k\times m}$. In other words the shift polynomial matrices are $D(q^{-1}) \in \mathbb{R}(q^{-1})^{m\times m}$ and $N(q^{-1}) \in \mathbb{R}(q^{-1})^{k\times m}$.

This representation gives an obvious way to define a candidate for a flat output ($z(t) \in \mathbb{R}^m$).

$$z(t) = S(q^{-1})y(t) + R(q^{-1})u(t). \tag{15}$$

where S and R satisfy, due to coprimeness of A and B, Bezout's matrix equation

$$\underbrace{R(q^{-1})}_{m\times m}\underbrace{D(q^{-1})}_{m\times m} + \underbrace{S(q^{-1})}_{m\times k}\underbrace{N(q^{-1})}_{k\times m} = I_{m\times m}, \tag{16}$$

where the new shift polynomial matrices are $R(q^{-1}) \in \mathbb{R}(q^{-1})^{m\times m}$ and $S(q^{-1}) \in \mathbb{R}(q^{-1})^{m\times k}$. Without going into details it can be shown, by keeping in mind the matrix nature of the polynomial matrices, that

$$u(t) = D(q^{-1})z(t) \tag{17}$$

$$y(t) = N(q^{-1})z(t). \tag{18}$$

Consequently, the equations (15), (17), and (18) can be used for the trajectory design for the output y by starting from a feasibly chosen flat output z.

It seems to be the case that a proper rational transfer matrix having an irreducible right coprime fraction representation (12) is differencely flat. So, we present the following result in the form of a conjecture, because all the details have not yet been verified.

Conjecture. The multivariable input-output system presented in an irreducible right coprime fraction form

$$y(t) = N(q^{-1})D(q^{-1})^{-1}u(t) \tag{19}$$

is differencely flat when represented in a nonreducible (minimal) observable and controllable state variable form

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) && (20)\\ y(t) &= Cx(t) && (21) \end{aligned}$$

Outline of Proof. The equation (17) is clear. From the equation (15) the output y and its delayed values can be eliminated by using the time-reversed state equation. This results in the desired form

$$z(t) = Vx(t) + W(q^{-1})u(t), \tag{22}$$

where $V \in \mathbb{R}^{m\times n}$, $x(t) \in \mathbb{R}^n$, and $W(q^{-1}) \in \mathbb{R}(q^{-1})^{m\times m}$. It seems that the idea of constructing for each row $T_i(q^{-1})$, corresponding to each output y_i, of the transfer matrix

$$T(q^{-1}) = \begin{bmatrix} T_1(q^{-1}) \\ T_2(q^{-1}) \\ \vdots \\ T_k(q^{-1}) \end{bmatrix}$$

a state variable representation, with appropriate subsystem matrices F_i, G_i, and C_i, of the form

$$\begin{aligned} x^i(t+1) &= F_i x^i(t) + G_i u(t)\\ y_i(t) &= C_i x^i(t) \end{aligned}$$

according to Chen [5], p. 265, is fruitful. Then along the same lines as in the SISO case one could construct for the state components $x^i_j(t)$ representations of the form

$$x^i_j(t) = \Lambda^i_j(q^{-1})z(t).$$

which then form a long vector related to the flat output z by

$$x(t) = \Lambda(q^{-1})z(t).$$

4. NONLINEAR SYSTEMS

Local equivalence of a nonlinear system with a linear one has in continuous-time framework studied by using a so-called Brunowský canonical form. It is simply a set of subsequent integrators driven by a single input. The number of the integrator sets is equal to the dimension of the control. The numbers of the integrators in each chain are so-called controllability indices, *c.f.* Chen [5], p. 190. In discrete-time systems the canonical form corresponding to the Brunowský one is called a *prime system*. It is a set of forward shift lines, each of which is driven by a separate input, see Aranda-Bricaire *et al.* [2], [3], & Marino *et al.* [17]. Consequently, the numbers of the forward shifts are the controllability indices, too.

Discrete-time systems are obtained via sampling of a correspondind continuous-time ones with the goal of obtaining a discrete model amenable for numerical calculations. Then the zeros of the sampled system may cause problems in control design. Inclusion of sampled systems to a general framework of discrete-time systems has in this respect studied in Monaco & Normand-Cyrot [18], [19]. Feedback linearisation was studied in [14]. On the other hand, open-loop control design based on flatness avoids these nonminimum phase problems. A comprehensive framework for studying nonlinear discrete-time systems was developed by Grizzle [15].

Theorems including linearizability via static diffeomorphisms or via state feedback are all valid for demonstrating the flatness of the original nonlinear system. Quite generally, without presenting detailed conditions or giving a proof, it can be stated the following.

Theorem. If the nonlinear discrete-time system

$$x(t+1) = f(x(t), u(t)) \tag{23}$$

is linearizable to a controllable linear system

$$\bar{x}(t+1) = F\bar{x}(t) + Gv(t)$$

via (sufficiently differentiable) transformations (*c.f.* Jakubczyk [16])

$$\bar{x} = \Phi(x); \quad u = k(x, v)$$

i.e. via the diffeomorphism Φ and the state feedback k then the system (23) is differencely flat.

5. EXAMPLE

A scalar discrete-time second order system

$$\begin{aligned} &y(t) + a_1 y(t-1) + a_2 y(t-2) = \\ &b_1 u(t-1) + b_2 u(t-2) \end{aligned} \tag{24}$$

is studied. Application of the polynomials $A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2}$ and $B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2}$ in the Bezouz's identity gives the first order polynomials

$$R(q^{-1}) = r_0 + r_1 q^{-1};\ S(q^{-1}) = s_0 + s_1 q^{-1}$$

The explicit flatness equations between the variables are then

$$\begin{aligned} z(t) &= s_0 y(t) + s_1 y(t-1) + r_0 u(t) + r_1 u(t-1)\\ u(t) &= z(t) + a_1 z(t-1) + a_2 z(t-2) \qquad (25)\\ y(t) &= b_1 z(t-1) + b_2 z(t-2) \end{aligned}$$

If the goal now is to drive the output y from 0 to $\bar{y}\,(\neq 0)$ as quickly as possible by suitably manipulating the control variable u, we have a dead-beat control problem. It can be shown that the minimal number of time steps required $N_{min} = \mathtt{deg}B^*(q) + 1$, where $\mathtt{deg}$ denotes the degree of the reciprocal polynomial $B^*(q) = q^{\mathtt{deg}A}B(q^{-1})$. In the model (24) $N_{min} = 2$. The design starts by choosing a step change for the flat output z:

$$z(t) = \begin{cases} 0, & t \leq 0 \\ \bar{z}, & 1 \leq t \end{cases}$$

The the input obtained form (25)

$$\begin{cases} u(0) &= 0 \\ u(1) &= \bar{z} \\ u(2) &= (1 + a_1)\bar{z} \\ u(t) &= (1 + a_1 + a_2)\bar{z}; \quad t = 3, 4, \ldots \end{cases}$$

produces the dead-beat output, *i.e.*

$$\begin{cases} y(1) &= 0 \\ y(2) &= b_1\bar{z} \\ y(t) &= \bar{y} = (b_1 + b_2)\bar{z}; \quad t = 3, 4, \ldots \end{cases}$$

If we want a smoother transfer of the output y from 0 to $\bar{y}$ then a gradual change of the flat output z form 0 to $\bar{z}$ can be applied via some intermediate values 0, $\bar{z}_1$, $\bar{z}_2$, $\ldots$, $\bar{z}_n$ giving the corresponding smoother control and output.

6. CONCLUDING REMARKS

The concept of flatness in discrete-time systems facilitates control design for dynamic systems via so-called flat output variables. A typical problem encountered in traditional control design is the non-minimum phase problem. Then controlled systems may become unstable. Design via flatness is independent of this property. On the other hand direct design gives the control only in open-loop mode. A conversion to practical closed-loop mode can be carried out via the flatness relations (8)-(11), *c.f.* [13]. Then the flatness-based control can be applied also under model uncertainties. Multivariable extensions work analogously to the example above. A corresponding nonlinear scalar study was reported in [4].

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