Polynomial-Predictive FIR Design – A Review

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ABSTRACT

In this paper, polynomial-predictive FIR (PPF) design is reviewed. Step-by-step instructions are given starting from the signal model, up to designing ideal fixed-point PPFs. This paper is a one-stop starting point for immediate application of PPFs. Also, a literature review is given, including examples of analogously designable filter types.

1. INTRODUCTION

Most real world signals exhibit smooth behavior, if adequately sampled, and the achieved noise level is sufficiently low. With smooth signals, piecewise polynomial signal model can be employed. Polynomial signals offer themselves for efficient prediction with polynomial-predictive FIRs (PPFs) [1]. PPFs, and their more sophisticated augmented versions, find applications in control field, for example, where they can be advantageously applied in fighting control loop delay.

Derivation of PPFs is reviewed in Section 2, and designing for exact polynomial prediction in fixed-point environments in Section 3. Magnitude response shaping feedback design is reviewed in Section 4. A literature review is given in Section 5, and Section 6 concludes the paper.

2. POLYNOMIAL-PREDICTIVE FIR DESIGN

The goal of PPF design [1] is to design such FIR coefficients $h(k)$, $k = 1, 2, ..., m$, where m is FIR length, that a piecewise polynomial input signal is exactly predicted. Thereafter, noise gain (1) of the FIR is minimized.

$$NG = \sum_{k=1}^{m} h(k)^2 \tag{1}$$

2.1. Derivation of Constraints for FIR Coefficients

Here, constraints on the filters coefficients are derived for $(p+1)$-steps-ahead PPFs (e.g. $p = 1$ yields two-steps-ahead prediction). With the latest input sample taken at time $n-1$, prediction of an input signal $x(n)$ is generally given by

$$x(n+p) = \sum_{k=1}^{m} h(k)x(n-k). \tag{2}$$

Polynomial signal model yields constraints on the PPF coefficients for each polynomial degree i up to the maximum input polynomial degree I.

0th degree constraint: Prediction of a constant signal $x(n) = c$ yields the constant c itself, and constraint g_0 as

$$\sum_{k=1}^{N} h(k)c = c \Leftrightarrow \sum_{k=1}^{m} h(k) = 1, \tag{3}$$

$$g_0 = \sum_{k=1}^{m} h(k) - 1 = 0. \tag{4}$$

1st degree constraint: prediction of a ramp signal $x(n) = an$ with a slope a, is given by

$$\sum_{k=1}^{m} h(k)a(n-k) = a(n+p) \Leftrightarrow \tag{5}$$

$$\sum_{k=1}^{m} h(k)(n-k) = n+p \Leftrightarrow \sum_{k=1}^{m} h(k)n - \sum_{k=1}^{m} h(k)k = n+p \tag{6}$$

$$\Leftrightarrow \sum_{k=1}^{m} h(k)n - \sum_{k=1}^{m} h(k)k = n+p \tag{7}$$

$$\Leftrightarrow \sum_{k=1}^{m} (h(k)-1)n = \sum_{k=1}^{m} h(k)k + p, \tag{8}$$

which, with (4), yields the 1st degree constraint g_1 as

$$g_1 = \sum_{k=1}^{m} h(k)k + p = 0. \tag{9}$$

2nd degree constraint: prediction of a parabola $x(n) = an^2$ is given by

$$\sum_{k=1}^{m} h(k)a(n-k)^2 = a(n+p)^2 \tag{10}$$

$$\Leftrightarrow \sum_{k=1}^{m} h(k)(n^2 - 2nk + k^2) = n^2 + 2np + p^2 \tag{11}$$

$$\Leftrightarrow \sum_{k=1}^{m} h(k)n^2 - 2\sum_{k=1}^{m} h(k)nk + \sum_{k=1}^{m} h(k)k^2 = n^2 + 2np + p^2 \tag{12}$$

$$\Leftrightarrow \sum_{k=1}^{m} (h(k)-1)n^2 - 2n\sum_{k=1}^{m} h(k)k + \sum_{k=1}^{m} h(k)k^2 = p^2. \tag{13}$$

With (4) and (9), (13) yields the 2nd degree constraint g_2.

$$g_2 = \sum_{k=1}^{m} h(k)k^2 - p^2 = 0 \tag{14}$$

In [1], the constraint for one-step-ahead prediction $p = 0$ of an Ith degree polynomial input signal is given as

$$g_I = \sum_{k=1}^{m} h(k)k^I = 0\,. \tag{15}$$

2.2. Derivation of FIR Coefficients from Their Constraints

To obtain closed form presentations for the filter coefficients from the constraints, method of Lagrange multipliers [2][1] may be applied. The objective is to minimize a function of the filter coefficients and Lagrange multipliers λ_i, $i = 1, 2, ..., I$, given by

$$L(h(1), h(2), \ldots, h(m), \lambda_0, \lambda_1, \ldots, \lambda_I) = NG + \lambda_0 g_0 + \lambda_1 g_1 + \cdots + \lambda_I g_I\,. \tag{16}$$

Function (16) includes the noise gain (1), and the constraints g_i, $i = 1, 2, ..., I$. (16) is minimized when its partial derivatives with respect to the filter coefficients and Lagrange multipliers are zero. For example, for the second degree PPFs ($I = 2$),

$$L = NG + \lambda_0 g_0 + \lambda_1 g_1 + \lambda_2 g_2 = \sum_{k=1}^{m} h(k)^2 + \lambda_0\left(\sum_{k=1}^{m} h(k) - 1\right) + \lambda_1\left(\sum_{k=1}^{m} h(k)k + p\right) + \lambda_2\left(\sum_{k=1}^{m} h(k)k^2 + p^2\right). \tag{17}$$

Next, the partial derivatives are calculated and set to zero:

$$\frac{\partial L}{\partial h(k)} = 2h(k) + \lambda_0 + \lambda_1 k + \lambda_2 k^2 = 0\,, \tag{18}$$

$$\frac{\partial L}{\partial \lambda_0} = \sum_{k=1}^{m} h(k) - 1 = 0\,, \tag{19}$$

$$\frac{\partial L}{\partial \lambda_1} = \sum_{k=1}^{m} h(k)k + p = 0\,, \tag{20}$$

$$\frac{\partial L}{\partial \lambda_2} = \sum_{k=1}^{m} h(k)k^2 - p^2 = 0\,. \tag{21}$$

The system (18)–(21) is solved by first solving $h(k)$ from (18), inserting $h(k)$ into (19)–(21) and solving for λ_0, λ_1, and λ_2, which are substituted back into (18), yielding the filter coefficients $h(k)$. Mathematica code for solving this system of equations is given in Fig. 1, along with the resulting closed form expression for $I = 2$ PPF coefficients for any p, and the coefficient values for $p = 1$, $m = 10$. Three lowest degree $p = 0$ PPFs are given below [1].

$$I = 0:\; h(k) = 1/m \tag{22}$$

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Solve[{Sum[(-(λ0 + λ1 k + λ2 k^2)/2), {k, 1, m}] - 1 == 0, Sum[(-k (λ0 + λ1 k + λ2 k^2)/2), {k, 1, m}] + p == 0,
  Sum[(-k^2 (λ0 + λ1 k + λ2 k^2)/2), {k, 1, m}] - p^2 == 0, hk == -(λ0 + k λ1 + λ2 k^2)/2}, {hk, λ0, λ1, λ2}]

{{hk → 30 k^2 (2 + 3 m + m^2 + 6 p + 6 m p + 6 p^2) / (m (-4 + m^2) (-1 + m^2)) + 3 (2 + 3 m + 3 m^2 + 6 p + 12 m p + 10 p^2) / ((-2 + m) (-1 + m) m) -
    (6 k (6 + 21 m + 21 m^2 + 6 m^3 + 22 p + 60 m p + 32 m^2 p + 30 p^2 + 30 m p^2)) /
    ((-1 + m) m (1 + m) (-4 + m^2)), λ0 → -6 (2 + 3 m + 3 m^2 + 6 p + 12 m p + 10 p^2) / ((-2 + m) (-1 + m) m),
  λ1 → 12 (6 + 21 m + 21 m^2 + 6 m^3 + 22 p + 60 m p + 32 m^2 p + 30 p^2 + 30 m p^2) / ((-1 + m) m (1 + m) (-4 + m^2)),
  λ2 → -60 (2 + 3 m + m^2 + 6 p + 6 m p + 6 p^2) / (m (-4 + m^2) (-1 + m^2))}}

% /. {m → 10, p → 1, k → {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}}

{{h{1,2,3,4,5,6,7,8,9,10} → {27/22, 19/30, 37/220, -37/220, -62/165, -5/11, -89/220, -149/660, 9/110, 57/110},
  λ0 → -39/10, λ1 → 1039/660, λ2 → -17/132}}
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Fig. 1. Solving the system of equations (18)–(21) with Mathematica. Also calculated are the values of the $p = 1$, $m = 10$, PPF coefficients. Notation: $h_k = h(k)$, $\lambda_i = \lambda(i)$.

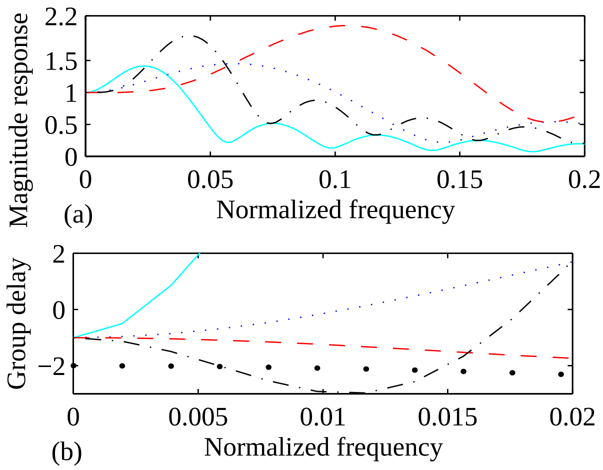

Fig. 2. Frequency responses (a) and group delays (b) of exemplary PPFs: $I = 1$, $p = 0$: $m = 20$ (dotted) and $m = 50$ (solid); $I = 2$, $p = 0$: $m = 20$ (dashed) and $m = 50$ (dash-dot). In (b), also the group delay of the PPF calculated in Fig. 1 (dark dotted).

$$I = 1:\; h(k) = (4m - 6k + 2)/[m(m-1)] \tag{23}$$

$$I = 2:\; h(k) = \frac{9m^2 + (9 - 36k)m + 30k^2 - 18k + 6}{m^3 - 3m^2 + 2m} \tag{24}$$

In (22)–(24), $k = 1, 2, ..., m$. In Fig. 2, exemplary PPF magnitude responses and group delays are shown. Also generally, passband peak is suppressed and the passband width gets narrower as PPF length increases or polynomial degree decreases. Prediction band, i.e., frequency range with group delay sufficiently close (depends on the application) to $-p-1$, gets narrower with increased PPF length or polynomial degree.

3. FIXED-POINT PPF DESIGN

Requirement for exact polynomial prediction in fixed-point environments is that the fixed-point PPF (FPPPF) coefficients $h_q(k)$ must exactly satisfy the constraints (4), (9), (14), and up to the constraint for degree I [3]. At simplest, FPPPF design can be an exhaustive search [4] over a limited region of a quantized coefficient space H, to find quantized coefficients $h_q(k) \in$ H, $k = 1, 2, ..., m$, which fulfill the constraints. An example of a search region spawning two quantization levels above and below the exact coefficients $h(k)$, given by (24), is seen in Fig. 3.

After finding all the sets of $h_q(k) \in$ H, $k = 1, 2, ..., m$, which exactly satisfy our constraints within the search region, the set that minimizes the noise gain (1) is selected as the ideally quantized coefficient PPF. One such set of coefficients is seen in Fig. 3. Though the ideally quantized coefficient PPFs make no assumptions of global noise gain minimization, their noise gain losses are negligible when compared with the noise gains of the corresponding non-quantized coefficient PPFs [3].

4. MAGNITUDE RESPONSE SHAPING

Magnitude responses of PPFs exhibit high inherent passband peaks, which is a drawback in most applications. By augmenting PPFs with appropriate feedbacks [5], PPF magnitude responses can be shaped without affecting the predictive properties. An augmented PPF of length $m = 2$ is illustrated in Fig. 4. This feedback has smoothing function, since the leftmost summation point yields a weighted sum of an input sample and its prediction, like also the second summation point within the delayline, as seen from Fig. 4. Thus, the feedback does not affect the predictive properties of the PPF. Conditions for the feedback to exactly preserve the predictive properties of the basis PPF are only that $\{b(k), 1 – b(k)\} \in$ H, $k = 1, 2, ..., m$, i.e., the feedback coefficients belong to the same quantized coefficient space H as the PPF coefficients $h(k)$ [3].

In Fig. 5, magnitude response and group delay of the shortest $I = 1$, $p = 0$, PPF (23), $m = 2$, is shown along with two augmented PPFs with the same basis PPF [3]. Using feedbacks, passband peak is clearly reduced. All filters in Fig. 5 are coefficient quantization error free with eight bit coefficients, i.e., they fulfill all the required constraints.

5. LITERATURE REVIEW

PPFs were derived by Heinonen and Neuvo in [1], where the case $p = 0$ is presented. Derivation of PPFs with any p is given in the appendix of [6], and in [7], least squares formulation of PPF design for any p and I is given in a matrix form suitable for numerical calculations. In [7], asymptotic noise gain of PPFs is also derived. In [8], a computationally efficient structure for implementing PPFs is given; complexity does not depend on the PPF length but only on I. In [9], PPFs [1] were formulated for complex-valued signals. Derivation and an example of an interpolated FIR approach to PPFs, resulting in lower complexity, is presented in [10]. To suppress passband peaks of PPFs, a prefiltering approach was proposed in [11]. Thereafter, *PPF feedback augmentation design* is given in [5]. In [5], examples are given on a modified first-degree PPF filter with a notch for power line frequency suppression, and on predictor-estimator cascade design providing for better stopband attenuation. Feedback augmentation is also presented in [12]. Finite coefficient word length is

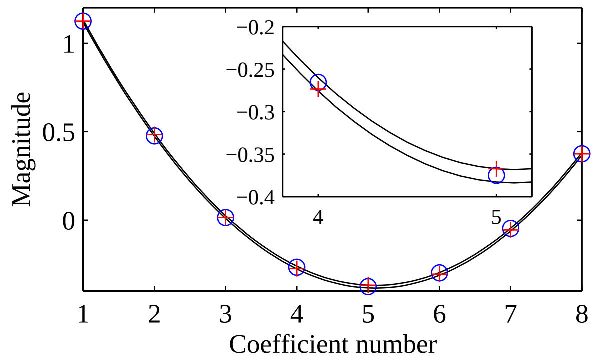

Fig. 3. A search band (between solid lines) for quantized coefficients of the $I = 2$, $p = 0$, $m = 8$, PPF with 8-bit coefficient precision. Circles ‘o’ denote the magnitude truncated, and plusses ‘+’ the ideally quantized coefficients.

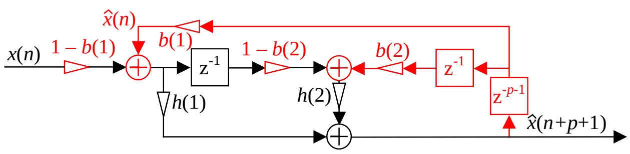

Fig. 4. Feedback augmented PPF of length $m = 2$. Hat denotes predictive estimate.

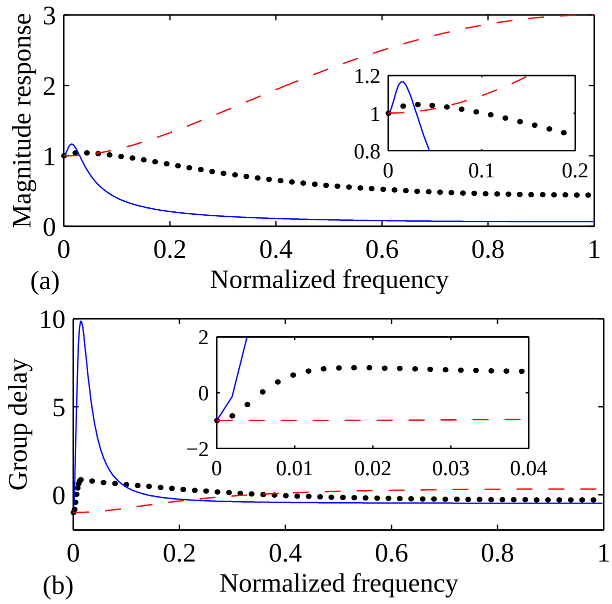

Fig. 5. Magnitude responses (a) and group delays (b) of augmented coefficient quantization error free $I = 1$, $p = 0$, $m = 2$ PPFs with the feedback coefficients $\{b(1), b(2)\} = \{0.6875, –0.9375\}$ (dark dotted) and $\{b(1), b(2)\} = \{0.9375, –0.9375\}$ (solid), along with their basis PPF $\{h(1), h(2)\} = \{2\ \ –1\}$ (dashed).

the topic of [13][14]. PPF feedback coefficients are optimized using a genetic algorithm in [15], and rigorously in [16]. *Design of exact fixed-point PPF implementations* is presented in [4] (also for predictive polynomial differentiator FIRs), along with integer programming interpretation of the design method. *Conditions for exact fixed-point implementations of predictive polynomial differentiator FIRs, and of their feedback augmentations*, are given in [3], where also a quantization error feedback approach for roundoff noise alleviation is proposed. PPFs are also the topic of [17]. Polynomial predictive filtering in control instrumentation is reviewed in [18] with several predictor structures. Matlab packages for PPF and augmented PPF design are available at [19] (note on the nomenclature: in [19] PPFs are referred to as Heinonen-Neuvo (H-N) filters, and augmentation can be done with the "FIR2IIR" designer).

Predictive polynomial differentiator FIRs (PPDFs) are derived analogously to PPFs; PPDFs for $I = 1$ are derived in [20], and for $I = 2$ in [21]; in them, also efficient recursive implementations are given. PPDFs get their feedback augmentations in [22]. PPDFs are also discussed in [23], and their coefficient quantization sensitivity is illustrated in [24]. PPDFs for angular acceleration measurement are reviewed in [25].

Sinusoidal predictors (SPs), also derived in time domain, are derived in [26] for application in power line frequency zero crossing detection. Evolved version of the system in [26] is presented in [27], where SPs are derived with constraints for suppression of DC and the first odd harmonic of the nominal SP frequency. IIR based computationally efficient implementation of SPs is given in [10]. A comprehensive presentation of SPs is given in [28], where also feedback augmentation is added to SPs.

6. CONCLUSIONS

Polynomial-predictive filtering, derived in time domain, is generally not well-known, and usually not mentioned in standard text books. Still, it is efficient and beneficial in processing many real-world signals. For most applications, it is sufficient to apply low degree polynomial predictors, i.e., of the maximum degree of $I = 1$, 2, or 3, which is also recommendable from the predictor magnitude response characteristics point of view.

In this paper, step-by-step instructions for designing polynomial-predictive filters are given, along with a fixed-point design method. A literature review on polynomial-predictive FIRs and associated time domain filters is given.

REFERENCES

[1] P. Heinonen and Y. Neuvo, "FIR-median hybrid filters with predictive FIR substructures," *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. 36, pp. 892–899, June 1988.

[2] D. Bertsekas, *Constrained Optimization and Lagrange Multipliers Methods.* New York, NY, USA: Academic Press, 1982.

[3] J. M. A. Tanskanen, "Coefficient quantization error free fixed-point IIR polynomial predictor design," in *Proc. 2000 IEEE Nordic Signal Processing Symp.*, Kolmården, Sweden, June 2000, pp. 219–222.

[4] J. M. A. Tanskanen and V. S. Dimitrov, "Round-off Error Free Fixed-Point Design of Polynomial FIR Predictors and Predictive FIR Differentiators," *Digital Signal Processing, A Review Journal*, accepted for publication. Also: Helsinki University of Technology Institute of Intelligent Power Electronics Publications, publication 4, Espoo, Finland, Aug. 2000 [online]. Available http://wooster.hut.fi/publications/tanskanen/round_off_error_free_predictors.pdf.

[5] S. J. Ovaska, O. Vainio, and T. I. Laakso, "Design of predictive IIR filters via feedback extension of FIR forward predictors," *IEEE Trans. Instrumentation and Measurement*, vol. 46, pp. 1196–1201, Oct. 1997.

[6] T. G. Campbell, "Design and implementation of image filters," Doctoral dissertation, Tampere University of Technology Publications 97, Tampere, Finland, 1992.

[7] P. Händel and P. Tichavský, "Asymptotic noise gain of polynomial predictors," *Signal Processing*, vol. 62, pp. 247–250, 1997.

[8] T. G. Campbell and Y. Neuvo, "Predictive FIR filters with low computational complexity," *IEEE Trans. Circuits and Systems*, vol. 38, pp. 1067–1071, Sept. 1991.

[9] T. Harju and T. I. Laakso, "Polynomial predictors for complex-valued vector signals," *Electronics Letters*, vol. 31, pp. 1650–1652, Sept. 1995.

[10] O. Vainio, "Design and efficient implementations of predictive FIR filters," *IEEE Trans. Instrumentation and Measurement*, vol. 44, pp. 864–868, Aug. 1995.

[11] T. I. Laakso and S. J. Ovaska, "Prefiltering approach for optimal polynomial prediction," *IEEE Trans. Signal Processing*, vol. 44, pp. 701–705, Mar. 1996.

[12] S. J. Ovaska and O. Vainio, "Predictive compensation of time-varying computing delay on real-time control systems," *IEEE Trans. Control Systems Tech.*, vol. 5, pp. 523–526, Sept. 1997.

[13] P. T. Harju, "Finite wordlength implementation of IIR polynomial predictive filters," in *Proc. 1997 IEEE Instrumentation and Measurement Tech. Conf.*, Ottawa, Canada, May 1997, pp. 60–65.

[14] P. T. Harju, "Roundoff noise properties of IIR polynomial predictive filters," in *Proc. 1997 IEEE Instrumentation and Measurement Tech. Conf.*, Ottawa, Canada, May 1997, pp. 66–71.

[15] P. T. Harju and S. J. Ovaska, "Optimization of polynomial predictive IIR filters using genetic algorithms," in *Proc. 3rd International Conf. on Signal Processing*, Beijing, China, Oct. 1996, pp. 68–71.

[16] P. T. Harju and S. J. Ovaska, "Optimization of IIR polynomial predictive filter magnitude response," *Signal Processing*, vol. 56, pp. 219–232, Feb. 1997.

[17] J. M. A. Tanskanen, "Polynomial predictive filters: implementation and applications," Doctoral dissertation, Helsinki University of Technology Institute of Intelligent Power Electronics Publications, Publication 5, Espoo, Finland, Nov. 2000.

[18] S. Väliviita, S. J. Ovaska, and O. Vainio, "Polynomial predictive filtering in control instrumentation: a review," *IEEE Trans. Industrial Electronics*, vol. 46, pp. 876–888, Oct. 1999.

[19] J. Martikainen, "Soft Filtering, an Emerging DSP Technique," [WWW Site]. Espoo, Finland: Helsinki University of Technology, Institute of Intelligent Power Electronics, Apr. 2000 [cited 9 Apr. 2001]. Available http://www.hut.fi/Units/PowerElectronics/soft/.

[20] O. Vainio, M. Renfors, and T. Saramäki, "Recursive implementation of FIR differentiators with optimum noise attenuation," *IEEE Trans. Instrumentation and Measurement*, vol. 46, pp. 1202–1207, Oct. 1997.

[21] S. Väliviita and O. Vainio, "Delayless differentiation algorithm and its efficient implementation for motion control applications," *IEEE Trans. Instrumentation and Measurement*, vol. 48, pp. 967–971, Oct. 1999.

[22] S. Väliviita and S. J. Ovaska, "Delayless recursive differentiator with efficient noise attenuation for control instrumentation," *Signal Processing*, vol. 69, pp. 267–280, Sept. 1998.

[23] S. Väliviita, "Predictive filtering methods for motor drive instrumentation," Doctoral dissertation, Helsinki University of Technology Institute of Intelligent Power Electronics Publications, publication 1, Espoo, Finland, Aug. 1998.

[24] J. M. A. Tanskanen and S. J. Ovaska, "Coefficient sensitivity of polynomial-predictive FIR differentiators: Analysis," in *Proc. 42nd IEEE Midwest Symp. on Circuits and Systems*, Las Cruces, NM, USA, Aug. 1999, pp. 405–408.

[25] S. J. Ovaska and S. Väliviita, "Angular acceleration measurement: a review," *IEEE Trans. Instrumentation and Measurement*, vol. 47, pp. 1211–1217, Oct. 1998.

[26] O. Vainio, "Noise reduction in zero crossing detection by predictive digital filtering," *IEEE Trans. Industrial Electronics*, vol. 42, pp. 58–62, Feb. 1995.

[27] O. Vainio and S. J. Ovaska, "Digital filtering for robust 50/60 Hz zero-crossing detectors," *IEEE Trans. Instrumentation and Measurement*, vol. 45, pp. 426–430, Apr. 1996.

[28] P. Händel, "Predictive digital filtering of sinusoidal signals," *IEEE Trans. Signal Processing*, vol. 46, pp. 364–374, Feb. 1998.