ADJUSTING THE GENERALIZED LIKELIHOOD RATIO TEST OF CIRCULARITY ROBUST TO NON-NORMALITY

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ABSTRACT

Recent research have elucidated that significant performance gains can be achieved by exploiting the circularity/non-circularity property of the complex-valued signals. The generalized likelihood ratio test (GLRT) of circularity [1, 2] assuming complex normal (Gaussian) sample has an asymptotic chi-squared distribution under the null hypothesis, but suffers from its sensitivity to Gaussianity assumption. With a slight adjustment, by diving the test statistic with an estimated scaled standardized 4th-order moment, the GLRT can be made asymptotically robust with respect to departures from Gaussianity within the wide-class of complex elliptically symmetric (CES) distributions while adhering to the same asymptotic chi-squared distribution. Our simulations demonstrate the validity of the χ^2 approximation even at small sample lengths. A practical communications example is provided to illustrate its applicability. In passing, we derive the connection with the kurtosis of a complex random variable with a CES distribution with the kurtosis of its real and imaginary part.

1. INTRODUCTION

Complex-valued signals are central in many communications, array signal processing and biomedical applications. Circularity property (or, properness) or lack of it (non-circularity) of the complex-valued signals can be exploited in designing wireless transceivers or array processors such as beamformers, DOA algorithms, blind source separation methods, etc. See [3, 4, 5, 6] to cite only a few. Circular random vector (r.v.) $\mathbf{z}$ of $\mathbb{C}^k$ has a vanishing pseudo-covariance matrix, namely, r.v. $\mathbf{z}$ is statistically uncorrelated with its complex-conjugate $\mathbf{z}^*$. For example, 4^k-QAM and 8-PSK modulated communications signals are circular but some other commonly used modulation schemes (such as BPSK or PAM) lead to non-circular signals. In case the signals or noise are non-circular, we need to take the full 2nd-order statistics into account when deriving or applying signal processing algorithms. Consequently optimal estimation and detection techniques and performance bounds are different for circular and non-circular cases. Hence, there needs to be a way to detect non-circularity.

We propose an improved version of the generalized likelihood ratio (GLR) test of circularity [1] (also studied in [2, 7, 8, 9]) derived under the assumption of complex normal sample. With a slight adjustment – by dividing the test statistic with an estimated scaled standardized 4th-order moment - the GLRT of circularity is made asymptotically robust with respect to violations of Gaussianity assumption. The proposed adjusted GLRT of circularity remains valid under the wide-class of complex elliptically symmetric (CES) distributions with finite 4th-order moments and adheres the same asymptotic chi-squared distribution under the null hypothesis.

The paper is organized as follows. Section 2 discusses the notion of circularity of complex r.v.'s while CES distributions are reviewed in Section 3. In Section 4, the kurtosis of a complex random variable (r.va.) with a CES distribution is linked with the kurtosis of its real and imaginary part, thus providing a deeper understanding on what the (complex) kurtosis actually measures. In Section 5, the GLRT of circularity is reviewed. Section 6 dwells on the adjusted GLRT test of circularity with particular attention paid on the estimation of the adjustment factor, the standardized 4th-order moment. In Section 7, simulation study demonstrates the high sensitivity of the GLRT of circularity to normality assumption whereas the adjusted GLRT is shown to work reliably. Also, the validity of the χ^2 approximation is demonstrated and a communications example is given to illustrate its usefulness.

2. ON CHARACTERIZATIONS OF CIRCULARITY

The distribution of a complex r.v $\mathbf{z} = \mathbf{x} + \jmath\mathbf{y} \in \mathbb{C}^k$ ($\jmath = \sqrt{-1}$ denoting the imaginary unit) is identified with the real $2k$-variate distribution of the composite real r.v. $\bar{\mathbf{z}} = (\mathbf{x}^T, \mathbf{y}^T)^T$ obtained by stacking the real part and imaginary part of $\mathbf{z}$. Thus $f(\mathbf{z})$ and $f(\bar{\mathbf{z}})$ are two alternative equivalent notations for the p.d.f. of $\mathbf{z}$. R.v. $\mathbf{z}$ is said to be *circular* if $\mathbf{z}$ has the same distribution as $e^{\jmath\theta}\mathbf{z}$ for all $\theta \in \mathbb{R}$. Naturally, the p.d.f $f(\mathbf{z})$ of a circular r.v. satisfies $f(e^{\jmath\theta}\mathbf{z}) = f(\mathbf{z})$ for all $\theta \in \mathbb{R}$.

We assume that r.v. $\mathbf{z}$ has mean zero and is non-degenerate

in any subspace of $\mathbb{C}^k$. Complete 2nd-order information of $\mathbf{z}$ is given by the $2k \times 2k$ real positive definite symmetric covariance matrix of the composite vector $\bar{\mathbf{z}}$,

$$E\big[\bar{\mathbf{z}}\bar{\mathbf{z}}^T\big] = \begin{pmatrix} E[\mathbf{x}\mathbf{x}^T] & E[\mathbf{x}\mathbf{y}^T] \\ E[\mathbf{y}\mathbf{x}^T] & E[\mathbf{y}\mathbf{y}^T] \end{pmatrix}. \tag{1}$$

Also the complex *covariance matrix* $\mathcal{C}(\mathbf{z}) \triangleq E\big[\mathbf{z}\mathbf{z}^H\big]$ (with $(\cdot)^H$ denoting the Hermitian transpose) and the complex *pseudo-covariance matrix* [10] $\mathcal{P}(\mathbf{z}) \triangleq E\big[\mathbf{z}\mathbf{z}^T\big]$ together contain the full 2nd-order information of a r.v. $\mathbf{z} \in \mathbb{C}^k$ since [11]

$$E[\mathbf{x}\mathbf{x}^T] = \tfrac{1}{2}\mathrm{Re}[\mathcal{C} + \mathcal{P}], \quad E[\mathbf{x}\mathbf{y}^T] = \tfrac{1}{2}\mathrm{Im}[-\mathcal{C} + \mathcal{P}], \tag{2}$$
$$E[\mathbf{y}\mathbf{y}^T] = \tfrac{1}{2}\mathrm{Re}[\mathcal{C} - \mathcal{P}], \quad E[\mathbf{y}\mathbf{x}^T] = \tfrac{1}{2}\mathrm{Im}[\mathcal{C} + \mathcal{P}]. \tag{3}$$

Note that $\mathcal{C}(\mathbf{z})$ is hermitian and positive definite $k \times k$ matrix and $\mathcal{P}(\mathbf{z})$ is symmetric complex $k \times k$ matrix. The inherent implication of circularity on the covariance structure of $\mathbf{z}$ is that its pseudo-covariance matrix vanishes, $\mathcal{P}(\mathbf{z}) = \mathbf{0}$, *i.e.*

$$E[\mathbf{x}\mathbf{x}^T] = E[\mathbf{y}\mathbf{y}^T] \ \text{ and } \ E[\mathbf{x}\mathbf{y}^T] = -E[\mathbf{y}\mathbf{x}^T]. \tag{4}$$

More generally, r.v. $\mathbf{z} \in \mathbb{C}^k$ with vanishing pseudo-covariance matrix is said to be 2*nd-order circular* [11] or *proper* [10].

Circularity coefficient [4]

$$r(z) \triangleq |\mathcal{P}(z)|/\mathcal{C}(z) \in [0, 1], \tag{5}$$

measures the "amount of circularity" of a zero mean complex r.va. $z = x + \jmath y$, where $\mathcal{C}(z) = E[|z|^2] > 0$ and $\mathcal{P}(z) = E[z^2] \in \mathbb{C}$ are the (co)variance and pseudo-(co)variance of z, respectively. In fact, $r(z)$ is the squared eccentricity of the ellipse defined by the real 2×2 covariance matrix (1) of $\bar{\mathbf{z}} = (x, y)^T \in \mathbb{R}^2$; See [7] for a comprehensive treatment of circularity coefficient. Specifically, the minimum 0 and the maximum 1 of circularity coefficient are of special interest:

$$r(z) = \begin{cases} 0, \text{iff } z \text{ is 2nd-order circular} \\ 1, \text{iff } x \text{ or } y \text{ is zero, or } x \text{ is a linear function of } y. \end{cases}$$

Note that $r(z) = 1$ if z is purely real-valued such as BPSK signal, or, if the signal lie on a line in the scatter plot (also called constellation or I/Q diagram) as is the case for BPSK, ASK, AM, or PAM-modulated communications signals.

3. CES DISTRIBUTIONS

R.v. $\mathbf{z} \in \mathbb{C}^k$ has k-variate *CN distribution* if the composite real r.v. $\bar{\mathbf{z}}$ has a $2k$-variate real normal distribution with covariances (2) and (3), denoted by $\mathbf{z} \sim \mathrm{CN}_k(\mathcal{C}, \mathcal{P})$. More generally, r.v. $\mathbf{z}$ has a (standardized) *CES distribution* if the composite real r.v. $\bar{\mathbf{z}}$ has a $2k$-variate real elliptically symmetric (RES) distribution with covariances (2) and (3) and a density generator g, the p.d.f. $f(\bar{\mathbf{z}})$ thus being proportional to

$$g\big(\bar{\mathbf{z}}^T E[\bar{\mathbf{z}}\bar{\mathbf{z}}^T]^{-1}\bar{\mathbf{z}}\big).$$

We write $\mathbf{z} \sim \mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g)$. See [12] for a comprehensive account on RES distributions. If (4) holds, *i.e.* $\mathcal{P} = \mathbf{0}$, then r.v. $\mathbf{z}$ has *circular CES distribution*, denoted $\mathbf{z} \sim \mathrm{CE}_k(\mathcal{C}, g)$ for short. The p.d.f. $f(\bar{\mathbf{z}})$ also has an intuitive complex representation in terms of $\mathbf{z}$ and $\mathcal{C}$ and $\mathcal{P}$ [11, 1].

CES distributions constitute a broad class of distributions that include the CN distribution as a particular special case when $g(\delta) = \exp(-\delta)$. The functional form of $g(\cdot)$ uniquely distinguishes among different CES distributions. Write

$$\mathcal{G}^q \triangleq \big\{ g : [0, \infty) \to [0, \infty) \ \big| \ \int_0^\infty t^{k+q-1}g(t)dt < \infty \big\}.$$

Then, any function $g \in \mathcal{G}^0$ is a valid density generator and $\mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g)$ for $g \in \mathcal{G}^q$ constitute the subclass of CES distributions with finite moments of order $2q$ (even moments).

Complex t-distribution with ν degrees of freedom ($0 < \nu < \infty$), labelled $T_{k,\nu}(\mathcal{C}, \mathcal{P})$, is obtained with $g_\nu(\delta) = (1 + 2\delta/(\nu - 2))^{-(2k+\nu)/2}$. Note that $g_\nu \in \mathcal{G}^2$ for $\nu > 4$. In the limit ($\nu \to \infty$), CN distribution is obtained. The larger the d.f. ν is, the closer the distribution is to CN distribution, whereas smaller valus for ν indicate increased heaviness of the tails. This property of $T_{k,\nu}$-distribution is useful for purposes of investigating the sensitivity of the GLRT to normality assumption later on in Section 7.

4. COMPLEX KURTOSIS EXPLAINED

Denote by

$$\gamma(x) \triangleq \frac{E[x^4]}{(E[x^2])^2}$$

the normalized 4th-order moment of a zero-mean real r.va. x. Recall that kurtosis, $\mathrm{kurt}_R(x) \triangleq \gamma(x) - 3$, is the shifted 4th-order moment with the property that $\mathrm{kurt}_R(x) = 0$ for a Gaussian r.va. In engineering literature, r.va.'s with negative (resp. positive) kurtosis are called *sub-Gaussian* (resp. *super-Gaussian*). If x has symmetric distribution then kurtosis measures peakedness (about the origin) combined with heaviness of the tails of the p.d.f. A typical super-Gaussian (resp. sub-Gaussian) r.va. has p.d.f. with a sharper peak and longer tails (resp. flatter peak and shorter tails).

There can be several different paths to generalize the notion of kurtosis for a complex r.va. z. Normalized 4th-order moment of a complex r.va. z can be defined as

$$\kappa(z) \triangleq \frac{E[|z|^4]}{(E[|z|^2])^2}. \tag{6}$$

Then the real-valued measure $\mathrm{kurt}_C(z) \triangleq \kappa(z) - r(z)^2 - 2$ is the most commonly used generalization of the kurtosis for a complex r.va. z (e.g. [5, 6, 9]). A complex Gaussian r.va. $z \sim \mathrm{CN}_1(\mathcal{C}, \mathcal{P})$ has kurtosis $\mathrm{kurt}_C(z) = 0$. It is not clear in the literature what the complex kurtosis really measures and a detailed study of its properties is still lacking. We now shed some light on this question.

If z has univariate ($k = 1$) CES distribution, $z = x+\jmath y \sim \mathrm{CE}_1(\mathcal{C}, \mathcal{P}, g)$, $g \in \mathcal{G}^2$, then $\gamma(x) = \gamma(y)$. Hence denote by γ (for short) the common 4th-order moment of x and y.

Theorem 1. *If* $z \sim \mathrm{CE}_1(\mathcal{C}, \mathcal{P}, g)$, $g \in \mathcal{G}^2$, *then*

$$\gamma = [3/(2 + r(z)^2)]\kappa(z)$$

where $r(z)$ *is the circularity coefficient* (5) *of* z *and* $\kappa(z)$ *the standardized 4th-order moment* (6) *of* z.

Proof. Since $\bar{\mathbf{z}} = (x, y)^T$ by definition has bivariate (non-singular) RES distribution, it can be represented as an invertible linear transform of a circular (spherical) r.v. $\bar{\mathbf{z}}_0 = (x_0, y_0)^T \in \mathbb{R}^2$ [12]. Thus, in complex form, $z = x + \jmath y$ has a stochastic representation:

$$z = az_0 + bz_0^*, \quad \text{for some } a, b \in \mathbb{C}, |a| \neq |b|,$$

where $z_0 = x_0 + \jmath y_0$ denotes the complex circular r.va. Hence $\mathcal{C} = E[|z|^2] = (|a|^2 + |b|^2)E[|z_0|^2]$ and $\mathcal{P} = E[z^2] = 2abE[|z_0|^2]$. Furthermore, $E[|z|^4]$ can be developed into

$$E[|z|^4] = (|a|^4 + |b|^4 + 4|a|^2|b|^2)E[|z_0|^4]$$
$$= (\mathcal{C}^2 + \tfrac{1}{2}|\mathcal{P}|^2)\kappa(z_0)$$

and hence $\kappa(z) = (1 + \tfrac{1}{2}r(z)^2)\kappa(z_0)$. Circularity of z_0 has severe implications on its moments, such as: $E[x_0^2] = E[y_0^2]$, $E[x_0 y_0] = 0$, $E[x_0^4] = E[y_0^4]$ and $E[x_0^2 y_0^2] = E[x_0^4]/3$. Thus, since $E[|z_0|^2] = 2E[x_0^2]$ and $E[|z_0|^4] = E[x_0^4] + E[y_0^4] + 2E[x_0^2 y_0^2]$, it follows that $\kappa(z_0) = \tfrac{2}{3}\gamma$, where $\gamma = E[x_0^4]/(E[x_0^2])^2 = E[x^4]/(E[x^2])^2$ denotes the common 4th-order moment of $x = \mathrm{Re}[z]$ and $y = \mathrm{Im}[z]$. This concludes the proof. $\square$

Theorem 1 also reveals the relationship of the real/imaginary kurtosis with the complex kurtosis:

$$\mathrm{kurt}_R(x) = \mathrm{kurt}_R(y) = [3/(2 + r(z)^2)]\mathrm{kurt}_C(z)$$

when $z \sim \mathrm{CE}_1(\mathcal{C}, \mathcal{P}, g)$, $g \in \mathcal{G}^2$. This means that the kurtosis of a CES distributed complex r.va. $z = x + \jmath y$ is simply a scaled version of the common kurtosis of its real and imaginary part. Hence it is instructive to inspect the scaling factor $[3/(2 + r(z)^2)]$ in more detail. If $r(z)$ attains the minimum 0 (*i.e.* z is 2nd-order circular), the scaling factor is $3/2$ (and $\mathrm{kurt}_C(z)$ reduces to $\mathrm{kurt}_C(z) = \kappa(z) - 2$). When $r(z)$ attains the maximum 1 (i.e. x or y is zero, or x is a linear function of y), the scaling factor is 1 and thus the real and the complex definitions of kurtosis coincide as expected. In the case that $r(z) \in (0, 1)$, the scaling factor takes values on the interval $(1, 3/2)$.

5. GLRT OF CIRCULARITY

Assume that $\mathbf{z}_1, \ldots, \mathbf{z}_n$ is i.i.d. random sample distributed as $\mathbf{z} \sim \mathrm{CN}_k(\mathcal{C}, \mathcal{P})$. In [1] (and independently in [2]), a GLR test

statistic was derived for the the hypothesis $H_0^N : \mathcal{P} = \mathbf{0}$ when $\mathbf{z} \sim \mathrm{CN}_k(\mathcal{C}, \mathcal{P})$ against the general alternative $H_1^N : \mathcal{P} \neq \mathbf{0}$. So the purpose is to test the validity of circularity assumption when sampling from CN distribution. GLR statistic is

$$q_n \triangleq \sup_{\mathcal{C}, \mathcal{P}} L_n(\mathcal{C}, \mathcal{P}) / \sup_{\mathcal{C}} L_n(\mathcal{C}, \mathbf{0}),$$

where $L_n(\mathcal{C}, \mathcal{P}) = \prod_{i=1}^n f(\mathbf{z}_i | \mathcal{C}, \mathcal{P})$ is the likelihood function of the sample $\mathbf{z}_1, \ldots, \mathbf{z}_n$ and $f(\cdot | \cdot, \cdot)$ the p.d.f. of the CN distribution. In [1], it was shown that

$$l_n \triangleq q_n^{-2/n} = \det(\mathbf{I} - \hat{\varrho}\hat{\varrho}^*) \tag{7}$$

where $\hat{\varrho} \triangleq \hat{\mathcal{C}}^{-1}\hat{\mathcal{P}}$ and $\hat{\mathcal{C}} \triangleq \mathrm{ave}\{\mathbf{z}_i \mathbf{z}_i^H\}$ and $\hat{\mathcal{P}} \triangleq \mathrm{ave}\{\mathbf{z}_i \mathbf{z}_i^T\}$ are the maximum likelihood estimators (MLE's) of $\mathcal{C}$ and $\mathcal{P}$ ("ave" stands for arithmetic mean over all indices of the argument; here $i = 1, \ldots, n$). This test statistic is invariant under the group of invertible linear transformations. The general likelihood ratio (LR) theory (e.g. [13]) readily attributes an asymptotic chi-squared distribution with $k(k + 1)$ degrees of freedom (d.f.) to the GLRT of circularity [7, 9, 8] .

Theorem 2. *Under* H_0^N, $-n\ln l_n \rightarrow \chi_p^2$ *in distribution, where* $p = k(k + 1)$ *and* l_n *is defined in* (7).

Above, the d.f. parameter p is the number of independent *real* parameters under the null hypothesis [13]. Since $\mathcal{P}$ is symmetric, null hypothesis involves $p' = k(k + 1)/2$ independent complex-valued parameters (namely, $\mathcal{P}_{ii}$ and $\mathcal{P}_{ij}$, for $i = 1, \ldots, k$ and $i < j$), and hence, $p = 2p' = k(k + 1)$ independent *real* parameters. In the scalar case ($k = 1$), the GLRT of circularity reduces to a test of sphericity [7, Sect. 4]. The test that rejects H_0^N whenever $-n\ln l_n$ exceeds the corresponding chi-square $(1 - \alpha)$th quantile is thus GLRT with asymptotic level α.

Normal-theory based LR-tests could often be used as tests of normality due to their vulnerability to normality assumption. This deficiency holds also for the GLRT of circularity (See Sect. 7.2). However, with a slight adjustment, more robust test can be derived.

6. ADJUSTED GLRT OF CIRCULARITY

Assume that $\mathbf{z}_1, \ldots, \mathbf{z}_n$ is i.i.d. random sample distributed as $\mathbf{z} \sim \mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g)$, $g \in \mathcal{G}^2$ and consider the hypothesis

$$H_0 : \mathcal{P} = \mathbf{0} \text{ when } \mathbf{z} \sim \mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g) \text{ with } g \in \mathcal{G}^2$$

against the alternative $H_1 : \mathcal{P} \neq 0$. Hence the purpose is to test the validity of circularity assumption when sampling from an unspecified (not necessarily normal) CES distribution with finite 4th-order moments. If $\mathbf{z} = \mathbf{x} + \jmath \mathbf{y} \sim \mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g)$, one has that (due to a well-known property of the marginals of a RES distributions)

$$\gamma(x_j) = \gamma(y_j), \forall j = 1, \ldots, k, \ \ \gamma(x_1) = \cdots = \gamma(x_k). \tag{8}$$

Hence, let γ denote value of the common normalized 4th-order moment when $\mathbf{z} \sim \mathrm{CE}_k(\mathcal{C}, \mathcal{P}, g)$, $g \in \mathcal{G}^2$.

Theorem 3. *Under H_0, $-n \ln l_n \to (\gamma/3)\chi_p^2$ in distribution.*

Proof. Follows from Corollary 1 of [13, pp. 415]. $\qquad \square$

Note that $\gamma = 3$ if $\mathbf{z} \sim \mathrm{CN}_k(\mathcal{C}, \mathcal{P})$ and hence Theorem 3 reduces to Theorem 2 as expected. Theorem 3 implies that by a slight adjustment, *i.e.* by dividing the GLRT-statistic $-n \log l_n$ by $\hat{\gamma}/3$, where $\hat{\gamma}$ is taken to be any consistent estimate of γ, we obtain an *adjusted GLR test statistic* of circularity,

$$\ell_n \triangleq -3(n-k) \ln l_n / \hat{\gamma}, \qquad (9)$$

that is valid – not just at the CN distribution, but – over the whole class of CES distributions with finite 4th-order moments. Note that we have replaced n by $(n-k)$ in defining the adjusted GLR test statistic ℓ_n. This follows from the Box approximation of the GLRT of circularity that tends to give better results for small n; see [8, Sect. VII-B]. Based on the asymptotic distribution, we reject the null hypothesis at (asymptotic) α-level if $p = 1 - F_{\chi_p^2}(\ell_n) < \alpha$.

The adjusted GLR test statistic requires a consistent estimate of γ. For the adjusted test to work properly, it is crucial to obtain an accurate estimate of γ. Based on (8), the *conventional estimate of γ* is $\hat{\gamma}_{con} \triangleq \mathrm{ave}\{\hat{\gamma}_j^x + \hat{\gamma}_j^y\}$, where $\hat{\gamma}_j^x$ and $\hat{\gamma}_j^y$ are the sample estimates of the 4th-order moments $\gamma(x_j)$ and $\gamma(y_j)$ of the marginal variables $x_j = \mathrm{Re}[z_j]$ and $y_j = \mathrm{Im}[z_j]$ $(j = 1, \ldots, k)$. Our investigations show that more accurate estimate of γ is obtained by utilizing Theorem 1 and the fact that $z_j \sim \mathrm{CE}_1(\mathcal{C}_{jj}, \mathcal{P}_{jj}, g)$, $j = 1, \ldots, k$. Due to (8) and Theorem 1, $\gamma = [3/(2+r(z_j)^2)]\kappa(z_j)$, $\forall j = 1, \ldots, k$. Hence we propose to calculate the estimate γ as the value

$$\hat{\gamma} = 3 \, \mathrm{ave}\{\hat{\kappa}_j / (2 + \hat{r}_j^2)\}, \qquad (10)$$

where $\hat{\kappa}_j$ (resp. $\hat{r}_j$) is the sample estimate of the 4th-order moment $\kappa(z_j)$ (resp. $r(z_j)$) of the jth marginal variable $z_j \in \mathbb{C}$ $(j = 1, \ldots, k)$ based on the sample $\mathbf{z}_1, \ldots, \mathbf{z}_n$. Simulations demonstrate the higher accuracy (smaller bias) of (10) over the conventional estimate $\hat{\gamma}_{con}$ (see Section 7.2). Since $r(z_j) = 0$ under the null hypothesis, we used $\hat{\gamma} = \frac{3}{2}\mathrm{ave}\{\hat{\kappa}_j\}$ as the estimate of γ in [9], but (10) is preferred.

7. SIMULATION STUDIES

7.1. On the validity of χ_p^2 approximation

We now investigate the validity of the χ_p^2 approximation to the finite sample distribution of the adjusted GLR test statistic ℓ_n in (9) at small sample lengths graphically via "chi-square plots". For this purpose, let $\ell_{n,1}, \ldots, \ell_{n,N}$ denote the computed values of ℓ_n from N simulated samples of length n and let $\ell_{n,[1]} \leq \cdots \leq \ell_{n,[N]}$ denote the ordered sample, *i.e.* the sample quantiles. Then $q_{[j]} = F_{\chi_p^2}^{-1}((j - 0.5)/N)$,
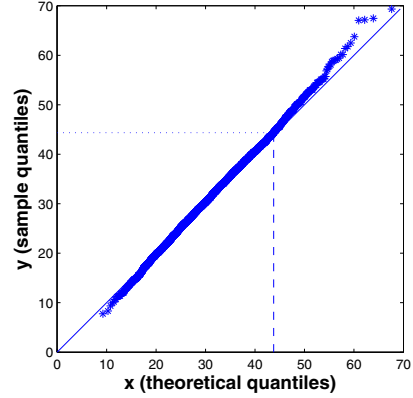

Fig. 1. Chi-square plot when sampling from $T_{k,\nu}(\mathcal{C})$ distribution ($n = 100$, $\nu = 6$, $k = 5$). The vertical (resp. horizontal) line indicate the value of 0.05-upper quantile.

$j = 1, \ldots, N$, are the corresponding theoretical quantiles (where 0.5 in $(j - 0.5)/N$ is a commonly used "continuity correction"). Then a plot of the points $(q_{[j]}, \ell_{n,[j]})$ should resemble a straight line through the origin having slope 1. Particularly, the theoretical $(1 - \alpha)$th quantile should be close to the corresponding sample quantile (e.g. $\alpha = 0.05$). Figure 1 depicts such chi-square plots when sampling from circular $T_{k,\nu}(\mathcal{C})$ distribution (with $k = 5$, $\nu = 6$) using sample length $n = 100$. Number of samples was $N = 5000$. We observe that a very good fit to the straight line is obtained even though the sample length is small. Similar result (not shown) are obtained for other choices of dimension k and degrees of freedom ν. The dashed vertical (resp. dotted horizontal) line indicate the value for the theoretical (resp. sample) 0.05-upper quantile. Quantiles are almost identical since the lines are crossing approximately on the diagonal.

In generating a random sample $T_{k,\nu}(\mathcal{C})$, we used that $\mathbf{z} = \sqrt{\nu - 2} \, \mathbf{z}_0 / \sqrt{s} \sim T_{k,\nu}(\mathbf{I})$ for independent r.v. $\mathbf{z}_0 \sim \mathrm{CN}_k(\mathbf{I})$ and r.va. $s \sim \chi_\nu^2$. Consequently, $\mathbf{z}' = \mathbf{Gz} \sim T_{k,\nu}(\mathcal{C})$ with covariance $\mathcal{C} = \mathbf{GG}^H$ for any non-singular $\mathbf{G} \in \mathbb{C}^{k \times k}$. The statistic l_n in (7) is invariant under invertible linear transformations of the data, but the accuracy of $\hat{\gamma}$ may depend on the value of $\mathcal{C}$. Hence, for the sake of fair comparison, matrix $\mathbf{G}$ was randomly generated for each trial, and hence each sample had a different value of the population covariance matrix $\mathcal{C}$.

7.2. Sensitivity of GLRT to violations of normality

Consider the case that the sample comes from $T_{k,\nu}(\mathcal{C})$ distribution. Hence the assumption of circularity of the sample is not violated (e.g. $\mathcal{P} = \mathbf{0}$), but the assumption of normality (H_0^N) is. The common standardized 4th-order moment then is $\gamma = 3(\nu - 2)/(\nu - 4)$. Figure 2 shows, based on $N = 5000$ Monte-Carlo trials, that the null hypothesis of circularity was falsely rejected (type I error, probability of false alarm) by GLR test at $\alpha = 0.05$ level in 87.3% of all trials when $\nu = 5$
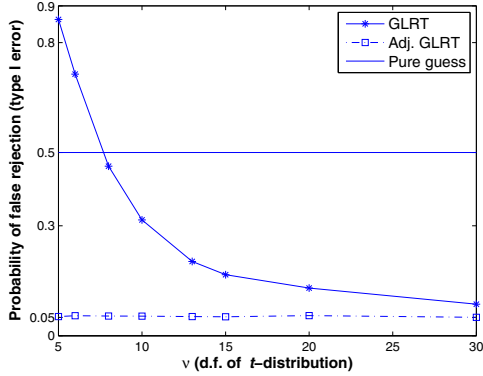
Fig. 2. Propability of falsely rejecting null when sampling from $T_{k,\nu}(\mathcal{C})$ distribution ($n = 500$, $k = 5$). See Sect. 7.2.

(and $\gamma = 9$). For $\nu = 30$ (and $\gamma = 3.23$), the probability of false alarm (PFA) was 10.6%. The adjusted GLRT of circularity stayed as expected close to the nominal $\alpha = 0.05$ level for all values of ν. For complex normal samples ($\nu = \infty$, $\gamma = 3$), PFA was 5.5% and 5.0% (not shown in the figure) for the GLR and the adjusted GLR test, respectively, which are very close to the α level. In our simulation, sample length was $n = 500$, dimension was $k = 5$, and covariance matrix $\mathcal{C}$ was different (randomly generated) for each trial.

We note that the proposed estimate (10) of γ results in a smaller bias as compared to the conventional estimate $\hat{\gamma}_{con}$. For example, when using the conventional estimate $\hat{\gamma}_{con}$, the PFA(%) of the adjusted GLRT were 7.3, 7.0, 6.3, 6.3, 6.0, 5.9, 6.0, 5.5 for $\nu = 5, 6, 8, 10, 13, 15, 20, 30$, respectively. These can be contrasted to PFA(%) values 5.2, 5.5, 5.4, 5.4, 5.2, 5.2, 5.5, 5.0 obtained using the estimate (10) of γ.

7.3. Communications example

Eight independent random circular signals - two QPSK signal, two 8-PSK two 32-QAM and two Gaussian signals - of equal power σ_s^2 are impinging on an $k = 12$ element uniform linear array (ULA) with a half a wavelength inter-element spacing from direction-of-arrivals (DOA's) that are randomly selected from $Unif(-90, 90)$ (degrees) for each trial. The noise $\mathbf{n}$ has circular CN distribution with $\mathcal{C}(\mathbf{n}) = \sigma_n^2 \mathbf{I}$. The signal to noise ratio (SNR) is $10 \log_{10}(\sigma_s^2/\sigma_n^2) = 20$dB and the number of snapshots is $n = 3000$. Since the noise and the sources are circular, the marginals z_i of the array output $\mathbf{z}$ are circular as well, so $\mathcal{P}(\mathbf{z}) = \mathbf{0}$. Then, based on 500 Monte-Carlo trials, the null hypothesis of (2nd-order) circularity was falsely rejected (type I error) by GLR test (resp. adjusted GLR test) at $\alpha = 0.05$ level in 2.0% (resp. 4.6%) of the trials. Hence both tests perform very well even though the Gaussian data assumption under which the GLRT test statistic l_n was derived do not hold exactly.

We further investigated the power of the GLRT and adjusted GLRT in detecting non-circularity. For this purpose,

we included a ninth source, a BPSK signal, that impinges on the array from a DOA randomly chosen from $Unif(-90, 90)$ degrees. Apart from this additional source signal, the simulation setting is exactly as earlier. Consequently, the array output $\mathbf{z}$ is no longer 2nd-order circular. The GLRT (resp. adjusted GLRT) correctly rejected at $\alpha = 0.05$ level the null hypothesis circularity for 98.7% (resp. 99.4%) of the trials.

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